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Model I – (M|M|1) : (∞ | ∞ or FCFS)
→ In steady state Condition
→ In transient state
→ infinite population.

* In steady state service discipline not allowed.

In this model we assume that arrival follows a Poisson distribution and service follows Poisson distribution/ Exponential distribution.

Here λ = mean arrival rate of units
μ = mean service rate.

To obtain the steady state Equation

Note: Here one event takes place in a small time interval Δt

There are $n>0$ units in the system at time $(t+\Delta t)$.

So there are three possiblities –

No. of Units at Time t		No. of Units at Time $(t+\Delta t)$
$(n-1)$	One arrival / No, service	(n) units
(n)	No, arrival / No. service	(n) units
$(n+1)$	No arrival / One service	(n) units

So, the probability $P_n(t+\Delta t)$ of the n units in the system at time $(t+\Delta t)$ is obtained

Ctd.

by adding the probabilities in the above three cases. {Note - 1. Each case are mutually Exclusive 2. Probabilities means independent probabilities}.

i.e.

$P_n(t+\Delta t) = P_{n-1}(t) \times$ Prob. of one arrival $\times$ Prob of No, Service

$+ P_n(t) \times$ Prob. of No service $\times$ Prob of. No arrival

$+ P_{n+1}(t) \times$ Prob. of one ~~arr~~ service $\times$ Prob. of. No arrival ——— (1)

Also we have.

One arrival in Δt. with probability $= \lambda \Delta t$ ——— (2)

No arrival in Δt with probability $= 1 - \lambda \Delta t$ ——— (3)

One service in Δt with probability $= \mu \Delta t$ ——— (4)

No service in Δt with probability $= 1 - \mu \Delta t$ ——— (5)

After using (2), (3), (4) & (5) we get the following from eq (1) -

$$P_n(t+\Delta t) = P_{n-1}(t) \times (\lambda \Delta t) \times (1-\mu \Delta t) + P_n(t) \times (1-\lambda \Delta t) \times (1-\mu \Delta t) + P_{n+1}(t) \times (\mu \Delta t) \times (1-\lambda \Delta t)$$

ctd—

$$p_n(t+\Delta t) = p_n(t)\left(1-(\lambda+\mu)\Delta t\right) + p_{n-1}(t)\,\lambda\Delta t + p_{n+1}(t)\,\mu\Delta t + \{p_{n-1}(t)\,\lambda\mu(\Delta t)^2\} + p_n(t)\,\lambda\mu(\Delta t)^2 + \{-p_{n+1}(t)\cdot\lambda\mu(\Delta t)^2\}$$

$$= p_n(t) - (\lambda+\mu)p_n(t)\Delta t + p_{n-1}(t)\lambda\Delta t + p_{n+1}(t)\mu\Delta t + O(\Delta t)$$

$$\Rightarrow \frac{p_n(t+\Delta t)-p_n(t)}{\Delta t} = \{-(\lambda+\mu)\}p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}$$

$$\Rightarrow \lim_{\Delta t\to 0}\left\{\frac{p_n(t+\Delta t)-p_n(t)}{\Delta t}\right\} = \lim_{\Delta t\to 0}\{(\lambda+\mu)p_n(t)\} + \lim_{\Delta t\to 0}\{\lambda p_{n-1}(t)\} + \lim_{\Delta t\to 0}\{\mu p_{n+1}(t)\} + \underbrace{\lim_{\Delta t\to 0}\frac{O(\Delta t)}{\Delta t}}_{0}$$

$$\Rightarrow \frac{dp_n(t)}{dt} = -(\lambda+\mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \quad \text{--- (6)}$$

where $n > 0$.

In the steady state —

$$\frac{dp_n(t)}{dt} \to 0, \quad \text{as } p_n(t) = P_n$$

So we get —

$$\boxed{0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}} \quad \text{--- (7)}$$

In the similar fashion, the probability that there is no unit in the system at time $(t+\Delta t)$, will be the sum of the following two probabilities —

Time t Units		Time $(t+\Delta t)$ Units
0	$\xrightarrow[\text{Service Not exist}]{\text{No Arrival}}$	0
1	$\xrightarrow[\text{One Service}]{\text{No Arrival}}$	0

i.e. $p_0(t+\Delta t) = p_0(t)(1-\lambda\Delta t) + p_1(t)(1-\lambda\Delta t)\mu\Delta t$

$$\Rightarrow \lim_{\Delta t\to 0}\left\{\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t}\right\} = \lim_{\Delta t\to 0}\left\{-\lambda p_0(t) + \mu p_1(t) + \frac{O(\Delta t)}{\Delta t}\right\}.$$

$$\Rightarrow \frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t) \quad \text{for } n=0. \quad (8)$$

Under steady state we get –

$$0 = -\lambda p_0 + \mu p_1 \quad \text{———} \quad (9)$$

Eq. (7) and Eq. (9) are called the steady state eqns of the system.

Now from eq. (9) we get

$p_1 = \left(\frac{\lambda}{\mu}\right)p_0$, Also here $\frac{\lambda}{\mu}<1$

$\vdots$

$p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0$, from (7)

$\vdots$

Generally we get

$$p_n = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

Also we know that $\sum_{n=0}^{\infty} p_n = 1$

$\Rightarrow p_0 + p_1 + \cdots + p_n + \cdots = 1$

$\Rightarrow p_0\left\{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^n + \cdots\right\} = 1$

$\Rightarrow p_0\left\{\frac{1}{1-\lambda/\mu}\right\} = 1$, $\frac{\lambda}{\mu}<1$ (As we have in this Model)

$p_0 = 1-\frac{\lambda}{\mu}$

As we know $\rho = \frac{\lambda}{\mu} = T.I$

$\boxed{p_0 = 1-\rho}$.

$p_n = \rho^n(1-\rho)$