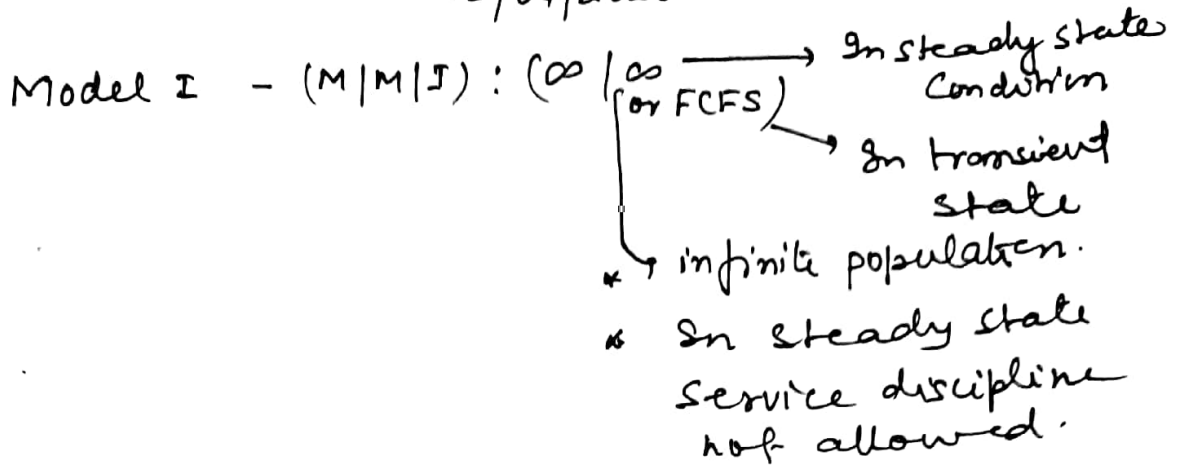


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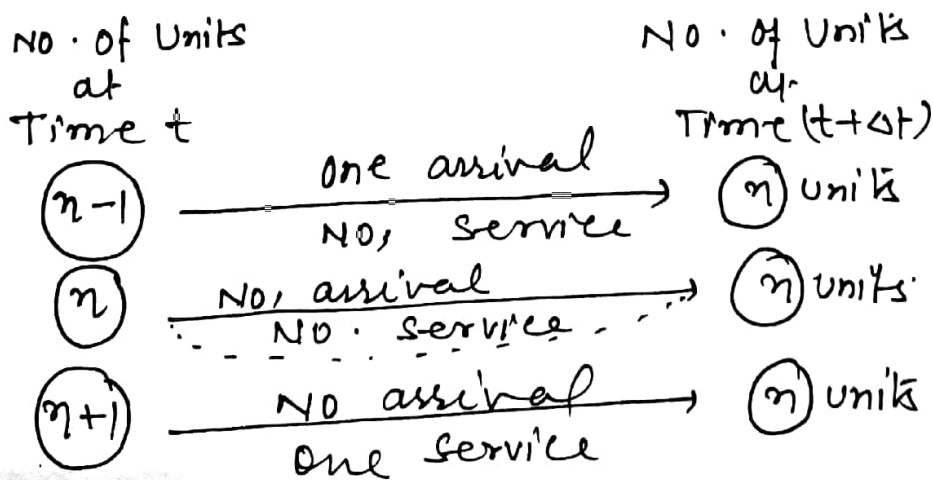


In this model we assume that arrival follows a Poisson distribution and service follows Poisson distribution / Exponential distribution.

Here  $\lambda$  = mean arrival rate of units  
 $\mu$  = mean service rate.

To obtain the steady state equation

Note: Here one event takes place in a small time interval  $\Delta t$  at time  $t$ . There are  $n > 0$  units in the system  $(t + \Delta t)$ . So there are three possibilities -



So, the probability  $P_n(t + \Delta t)$  of the  $n$  units in the system at time  $(t + \Delta t)$  is obtained as follows.

by adding the probabilities in the above three cases. ②

i.e.

$$\begin{aligned}
 P_m(t+\Delta t) = & P_{m-1}(t) \times \text{Prob. of one arrival} \times \text{Prob. of NO service} \\
 & + P_m(t) \times \text{Prob. of NO service} \times \text{Prob. of NO arrival} \\
 & + P_{m+1}(t) \times \text{Prob. of one service} \times \text{Prob. of NO arrival}
 \end{aligned}
 \tag{1}$$

{ NOTE -  
 1. Each case are mutually Exclusive  
 2. Probabilities means independent probabilities }

Also we have .

one arrival in  $\Delta t$  with probability  $= \lambda \Delta t$  ②

NO arrival in  $\Delta t$  with probability  $= 1 - \lambda \Delta t$  ③

One service in  $\Delta t$  with probability  $= \mu \Delta t$  ④

NO service in  $\Delta t$  with probability  $= 1 - \mu \Delta t$  ⑤

After using ②, ③, ④ & ⑤ we get the following from eq ① -

$$P_m(t+\Delta t) = P_{m-1}(t) \times (\lambda \Delta t) \times (1 - \mu \Delta t) + P_m(t) \times (1 - \lambda \Delta t) \times (1 - \mu \Delta t) + P_{m+1}(t) \times (\mu \Delta t) \times (1 - \lambda \Delta t)$$

chd

(3)

$$\begin{aligned}
 p_n(t+\Delta t) &= p_n(t)(1 - (\lambda + \mu)\Delta t) + p_{n-1}(t)\lambda\Delta t \\
 &\quad + p_{n+1}(t)\mu\Delta t + \underbrace{p_{n-1}(t)\lambda\mu(\Delta t)^2}_{\checkmark} \\
 &\quad + \underbrace{p_n(t)\lambda\mu(\Delta t)^2}_{\checkmark} + \underbrace{-p_{n+1}(t)\lambda\mu(\Delta t)^2}_{\checkmark} \\
 &= p_n(t) - (\lambda + \mu)p_n(t)\Delta t + p_{n-1}(t)\lambda\Delta t \\
 &\quad + p_{n+1}(t)\mu\Delta t + O(\Delta t)
 \end{aligned}$$

$$\Rightarrow \frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} = \underbrace{-(\lambda + \mu)p_n(t)}_{\checkmark} + \underbrace{\lambda p_{n-1}(t)}_{\checkmark} + \underbrace{\mu p_{n+1}(t)}_{\checkmark} + \frac{O(\Delta t)}{\Delta t}$$

$$\begin{aligned}
 \Rightarrow \lim_{\Delta t \rightarrow 0} \left\{ \frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} \right\} &= \lim_{\Delta t \rightarrow 0} \left\{ -(\lambda + \mu)p_n(t) \right\} + \lim_{\Delta t \rightarrow 0} \left\{ \lambda p_{n-1}(t) \right\} \\
 &\quad + \lim_{\Delta t \rightarrow 0} \left\{ \mu p_{n+1}(t) \right\} + \lim_{\Delta t \rightarrow 0} \frac{O(\Delta t)}{\Delta t} \\
 &\qquad\qquad\qquad \downarrow 0
 \end{aligned}$$

$$\Rightarrow \frac{dp_n(t)}{dt} = -(\lambda + \mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \quad \text{--- (6)}$$

where  $n > 0$ .

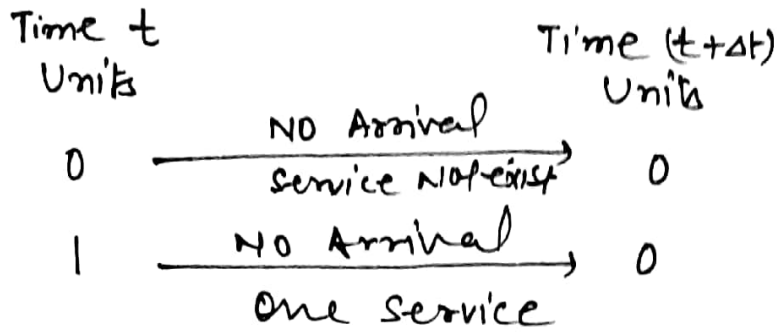
In the steady state —

$$\frac{dp_n(t)}{dt} \rightarrow 0, \text{ as } p_n(t) = P_n$$

So we get —

$$\boxed{0 = -(\lambda + \mu)P_n + \lambda P_{n-1} + \mu P_{n+1}} \quad \text{--- (7)}$$

In the similar fashion, the probability that there is no unit in the system at time  $(t+\Delta t)$ , will be the sum of the following two probabilities —



i.e.  $p_0(t+\Delta t) = p_0(t) (1-\lambda\Delta t) + p_1(t) (1-\lambda\Delta t)\mu\Delta t$

$$\Rightarrow \lim_{\Delta t \rightarrow 0} \left\{ \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} \right\} = \lim_{\Delta t \rightarrow 0} \left\{ -\lambda p_0(t) + \mu p_1(t) + \frac{0(\Delta t)}{\Delta t} \right\}$$

$$\Rightarrow \frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t) \quad \text{for } n=0. \text{ (8)}$$

Under steady state we get -

$$0 = -\lambda p_0 + \mu p_1 \quad \text{--- (9)}$$

Eq. (7) and Eq. (9) are called the steady state eq<sup>n</sup>s of the system.

Now from eq. (9) we get-

$$p_1 = \left(\frac{\lambda}{\mu}\right) p_0, \quad \text{Also here } \frac{\lambda}{\mu} < 1$$

$$p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0, \quad \text{from (7)}$$

Generally we get-

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$$

Also we know that  $\sum_{n=0}^{\infty} p_n = 1$

$$\Rightarrow p_0 + p_1 + \dots + p_n + \dots = 1$$

$$\Rightarrow p_0 \left\{ 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \dots + \left(\frac{\lambda}{\mu}\right)^n + \dots \right\} = 1$$

$$\Rightarrow p_0 \left\{ \frac{1}{1-\lambda/\mu} \right\} = 1, \quad \frac{\lambda}{\mu} < 1 \quad \left( \text{As we have in this model} \right)$$

$$p_0 = \frac{1-\lambda/\mu}{\mu}$$

As we know  $\rho = \frac{\lambda}{\mu} = T.R$

$$p_0 = 1 - \rho$$

$$p_n = \rho^n (1 - \rho)$$